A New Formulation in Lattice Theory

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Abstract

In this paper we propose a new approach to formulate the field theory on a lattice. This approach can eliminate the Fermion doubling problem, preserve the chiral symmetry and get the same dispersion relation for both Fermion and Boson fields. This gives us the possibility to write down the chiral model (such as the Weinberg-Salam model) on a lattice.

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It is well-known that the lattice theory suffers from the problem of Fermion doubling. To cure from this Wilson [1] [2] added a new term, so-called the "Wilson term", to the lattice Fermion Lagrangian. This term kills the superfluous components of the Fermion field but breaks the chiral symmetry and leads to different dispersion relations for Fermions and Bosons. Kogut and Susskind [3] [4] proposed the "stagger model", but it needs four generations of Fermions at least, and deals with the Fermions and Bosons on different footing. Besides these two popular prescriptions, Drell et al [5] developed another approach which preserves the chiral symmetry and at the same time correctly counts the number of Fermion states. Their crucial point was to introduce a lattice gradient operator which couples all lattice sites along a given direction instead of coupling only nearest-neighbor sites.

In this paper, we propose to adopt a new approach in which a new gradient operator, in some sence likes the one used by Drell et al, is introduced. By using our approach, we can eliminate the Fermion doubling problem and at the same time preserve the chiral symmetry.

Besides these, we can get the same dispersion relation for both Fermions and Bosons, which will be important if we want to construct the supersymmetry model on a lattice.

First let us take a look upon how the Fermion-doubling problem arises. For free scalar field $\phi(x)$, the lattice action in the Euclidean space is (for more detail, see [6])

$$\begin{aligned} S(\phi) &= \sum_n \{ \frac{a^2}{2} \sum_\mu (\phi_{n+\mu} - \phi_n)^2 + \frac{a^4}{2} m^2 \phi_n^2 \} \\ &= \sum_n \{ \frac{a^2}{2} \sum_\mu [(R_\mu - 1)\phi_n]^2 + \frac{a^4}{2} m^2 \phi_n^2 \}, \end{aligned} \tag{1}$$

with the definition of the displacement operator R_μ as

$$R_\mu \phi_n = \phi_{n+\mu}. \tag{2}$$

Using the formula

$$\phi_n = \int \frac{d^4k}{(2\pi)^4} e^{ikna} \phi(k), \qquad -\frac{\pi}{a} \leq k_\mu \leq \frac{\pi}{a}, \quad for \quad each \quad \mu, \tag{3}$$

we can transfer (1) into the momentum space form

$$S(\phi) = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \phi(-k) [\sum_\mu \frac{4}{a^2} \sin^2(\frac{ak_\mu}{2}) + m^2] \phi(k). \tag{4}$$

The dispersion relation read from this equation is

$$S_\phi(k) = \sum_\mu \frac{4}{a^2} \sin^2(\frac{ak_\mu}{2}) + m^2. \tag{5}$$

For Fermion field the lattice action is usually taken as

$$S(\psi) = \sum_n \{ \frac{a^3}{2} \sum_\mu \bar{\psi}_n \gamma_\mu (R_\mu - R_{-\mu}) \psi_n + m a^4 \bar{\psi}_n \psi_n \}. \tag{6}$$

Transfering it into momentum space we get

$$S(\psi) = \int \frac{d^4k}{(2\pi)^4} \bar{\psi}(-k) [i \sum_\mu \gamma_\mu \frac{\sin(ak_\mu)}{a} + m] \psi(k). \tag{7}$$

From this equation we see the dispersion relation

$$S_\psi(k) = \sum_\mu \frac{1}{a^2} \sin^2(ak_\mu) + m^2 \tag{8}$$

is different from (5), and a doubling of the Fermionic degrees of freedom appears.

Comparing (1) with (6) we can see that the difference results from the different definition of the gradient operator ∂_μ in bosonic and fermionic cases.

For scalar field we use the replacement

$$\partial_\mu \longrightarrow \frac{R_\mu - 1}{a}, \tag{9}$$

(hence yielding the D'Alembertian $\Box = \sum_\mu \frac{R_\mu + R_{-\mu} - 2}{a^2}$ in the lattice theory effectively). From some point of view, we can imagine the gradient operator used here means the right-traslation for one unit of lattice spacing a in direction μ. So it needs N times for walking all lattice sites in direction μ.

However for Fermion field to keeping the Hermitian of the action we use the replacement

$$\partial_\mu \longrightarrow \frac{R_\mu - R_{-\mu}}{2a}. \tag{10}$$

We can imagine the action of the gradient operator now means simultaneously translating the function from a given site to both the right and left ones by one unit of lattice spacing a in direction μ. So only $\frac{N}{2}$ times is needed for walking all points on the lattice. This different translation "speed" explains why there is double-counting in formula (6) but not in (4). It is evident that if one uses the latter choice of replacement (10) instead of the former one (9) in (1), there will be double-counting for Boson fields as well. So one way of solving the problem is to find another replacement of the operator ∂_μ , whose action will translate only one unit of lattice spacing at each step for Fermion field.

Consulting the definition of the translation operator in the momentun space

$$\begin{aligned} R_\mu \phi_n &= R_\mu [\sum_k e^{ikna} \psi(k)] \\ &= \phi_{n+\mu} = \sum_k e^{ikna} \psi(k) e^{ik_\mu a}, \end{aligned} \tag{11}$$

we can define an operator $R_{\frac{\mu}{2}}$, corresponding to the "half-spacing translation", as

$$R_{\frac{\mu}{2}}\psi_n := \sum_k e^{ikna}\psi(k)e^{\frac{ik_\mu a}{2}};$$
$$R_{-\frac{\mu}{2}}\psi_n := \sum_k e^{ikna}\psi(k)e^{-\frac{ik_\mu a}{2}}. \tag{12}$$

From the definition (12) we obtain

$$R^2_{\frac{\mu}{2}} = R_\mu, \tag{13}$$
$$R_{\frac{\mu}{2}}R_{-\frac{\mu}{2}} = R_{-\frac{\mu}{2}}R_{\frac{\mu}{2}} = I, \tag{14}$$
$$\Box = \frac{1}{a^2}\sum_\mu (R_{\frac{\mu}{2}} - R_{-\frac{\mu}{2}})^2 \tag{15}$$

Now using the replacement

$$\partial_\mu \longrightarrow \frac{R_{\frac{\mu}{2}} - R_{-\frac{\mu}{2}}}{a} \tag{16}$$

in the case of Fermion field and substituting it into (6), we obtain

$$S(\psi) = \sum_n \{a^3 \sum_\mu \bar{\psi}_n\gamma_\mu(R_{\frac{\mu}{2}} - R_{-\frac{\mu}{2}})\psi_n + ma^4\bar{\psi}_n\psi_n\} \tag{17}$$

and the momentum space form

$$S(\psi) = \int \frac{d^4k}{(2\pi)^4}\bar{\psi}(-k)[2i\sum_\mu \gamma_\mu \frac{\sin(\frac{ak_\mu}{2})}{a} + m]\psi(k) \tag{18}$$

From this we see immediately that there is no double-counting anymore and the Fermion field has the same dispersion relation as the one for scalar field (5). When $m = 0$ this new formula (17) will preserve the chiral symmetry.

From the definition (12) we can also see that $R_{\frac{\mu}{2}}$ and $R_{-\frac{\mu}{2}}$ are not "local operators". When acting on some functions of lattice sites, they not only concern with the nearest neighbor sites but also all the sites in direction μ on the lattice. (In this sence our choice is similar to the one adopted by Drell et al). However, it is evident that the combination $\frac{R_{\frac{\mu}{2}} - R_{-\frac{\mu}{2}}}{a}$ will approximate to ∂_μ when the lattice spacing $a \to 0$. So we can call the combination

$\frac{R_{\frac{\mu}{2}}-R_{-\frac{\mu}{2}}}{a}$ the " quasi-local operator". The quasi-local feature is the price for getting the above mentioned nice characters.

On the other hand, we emphasize that the gradient operator (16) we used here is essentially different from that used by Drell et al. Our operators $R_{\frac{\mu}{2}}, R_{-\frac{\mu}{2}}$ and ∂_μ are closely related to the " on-site" operators $R_\mu, R_{-\mu}$ *et al* through Eqs(13),(14) and (15), and share a number of common properties as the " on-site" operator (cf (19),(20)and (21)).

The next important work is to construct the lattice gauge theory in this new frame. For convenience, let's first list some useful relations as follows:

$$R_{\frac{\mu}{2}}(\Phi_n\psi_n) = (R_{\frac{\mu}{2}}\Phi_n)(R_{\frac{\mu}{2}}\psi_n), \tag{19}$$

$$R_{\frac{\mu}{2}}\psi_n = R_{-\frac{\mu}{2}}\psi_{n+\mu}, \tag{20}$$

$$[R_{\frac{\mu}{2}}\Phi_n]^+ = R_{\frac{\mu}{2}}\Phi_n^+ \tag{21}$$

and when $a \to 0$

$$R_\mu \sim 1 + a\partial_\mu, \qquad R_{\frac{\mu}{2}} \sim 1 + \frac{a}{2}\partial_\mu. \tag{22}$$

The gauge invariant action of Fermion field corresponding to (6) can be written as

$$S(\psi) = \sum_n \{a^3 \sum_\mu \bar{\psi}_n \gamma_\mu [U_{n,\frac{\mu}{2}} R_{\frac{\mu}{2}} - U_{n,-\frac{\mu}{2}} R_{-\frac{\mu}{2}})]\psi_n + ma^4\bar{\psi}_n\psi_n\}. \tag{23}$$

Here $U_{n,\frac{\mu}{2}}$ is the element of gauge group and can be written as

$$U_{n,\frac{\mu}{2}} = e^{i\frac{a}{2}gA_{n,\frac{\mu}{2}}} = exp\{i\frac{a}{2}g\frac{\lambda^i}{2}A^i_{n,\frac{\mu}{2}}\}. \tag{24}$$

We want to emphasize that the meaning of two indeces of U are essentially different: the first index n denotes the site of lattice point and the second index $\frac{\mu}{2}$ denotes the translation direction and the translation distance. So the operator R acts only on the first index (By this distinction, we can think $A_{n,\frac{\mu}{2}}$ as the field on lattice site n with an arrow along the $\frac{\mu}{2}$ direction, rather than the "link variable" in old language) . The local guage transforms of Fermion field and guage field are given by

$$\psi_n \longrightarrow \Phi_n \psi_n; \tag{25}$$

$$\begin{aligned} U_{n,\frac{\mu}{2}} &\longrightarrow \Phi_n U_{n,\frac{\mu}{2}} [R_{\frac{\mu}{2}} \Phi_n]^+, \\ U_{n,-\frac{\mu}{2}} &\longrightarrow \Phi_n U_{n,-\frac{\mu}{2}} [R_{-\frac{\mu}{2}} \Phi_n]^+ \end{aligned} \tag{26}$$

respectively. From (26) we can see the guage transform rule of $U_{n,-\frac{\mu}{2}}$ is same as that of $[R_{-\frac{\mu}{2}} U_{n,\frac{\mu}{2}}]^+$. So we consider them as the same quantity. Then by making use of the Eqs (19) (21) and the Hermitian of λ^i matrices we obtain

$$U_{n,-\frac{\mu}{2}} = exp\{-i\frac{a}{2} g R_{-\frac{\mu}{2}} A_{n,\frac{\mu}{2}}\}. \tag{27}$$

Now the action of guage field is taken as

$$S(A) = -\frac{8}{g^2} \sum_p trU_p, \quad p \in \quad all \quad plaquettes \tag{28}$$

where

$$U_p = U_{n,\frac{\mu}{2}} (R_{-\frac{\mu}{2}} U_{n+\mu,\frac{\nu}{2}})(R_{-\frac{\mu}{2}} R_{-\frac{\nu}{2}} U_{n+\mu+\nu,-\frac{\mu}{2}})(R_{-\frac{\nu}{2}} U_{n+\nu,-\frac{\nu}{2}}). \tag{29}$$

From (26) (20) (21) it is easy to see that U_p is gauge invariant. And the above equation can be simplified by using (27)

$$\begin{aligned} R_{-\frac{\mu}{2}} U_{n+\mu,\frac{\nu}{2}} &= R_{-\frac{\mu}{2}} exp\{i\frac{a}{2} g A_{n+\mu,\frac{\nu}{2}}\} \\ &= exp\{i\frac{a}{2} g R_{-\frac{\mu}{2}} A_{n+\mu,\frac{\nu}{2}}\} = exp\{i\frac{a}{2} g R_{\frac{\mu}{2}} A_{n,\frac{\nu}{2}}\}, \\ R_{-\frac{\mu}{2}} R_{-\frac{\nu}{2}} U_{n+\mu+\nu,-\frac{\mu}{2}} &= R_{-\frac{\mu}{2}} R_{-\frac{\nu}{2}} exp\{-i\frac{a}{2} g R_{-\frac{\mu}{2}} A_{n+\mu+\nu,\frac{\mu}{2}}\} \\ &= exp\{-i\frac{a}{2} g R_{\frac{\nu}{2}} A_{n,\frac{\mu}{2}}\}, \\ R_{-\frac{\nu}{2}} U_{n+\nu,-\frac{\nu}{2}} &= R_{-\frac{\nu}{2}} exp\{-i\frac{a}{2} g R_{-\frac{\nu}{2}} A_{n+\nu,\frac{\nu}{2}}\} \\ &= exp\{-i\frac{a}{2} g A_{n,\frac{\nu}{2}}\}, \end{aligned} \tag{30}$$

and then

$$U_p = exp\{i\frac{a}{2} g A_{n,\frac{\mu}{2}}\} \ exp\{i\frac{a}{2} g R_{\frac{\mu}{2}} A_{n,\frac{\nu}{2}}\} exp\{-i\frac{a}{2} g R_{\frac{\nu}{2}} A_{n,\frac{\mu}{2}}\} exp\{-i\frac{a}{2} g A_{n,\frac{\nu}{2}}\}. \tag{31}$$

From this expression it can be seen the sum of all "plaquettes" is equal to the sum of all different pairs (n, μ, ν).

Now we need to prove when $a \longrightarrow 0$, all the formulae we obtained above will come back to their corresponding continuum form. First we discuss the action $S(A)$. Substituting (22) into (31) we get

$$\begin{aligned} U_p &\sim exp\{i\frac{a}{2}gA_{n,\frac{\mu}{2}}\}exp\{i\frac{a}{2}g(A_{n,\frac{\nu}{2}}+\frac{a}{2}\partial_\mu A_{n,\frac{\nu}{2}})\} \\ &\quad exp\{-i\frac{a}{2}g(A_{n,\frac{\mu}{2}}+\frac{a}{2}\partial_\nu A_{n,\frac{\mu}{2}})\}exp\{-i\frac{a}{2}gA_{n,\frac{\nu}{2}}\} \\ &\sim exp\{i\frac{a^2g}{4}(\partial_\mu A_{n,\frac{\nu}{2}}-\partial_\nu A_{n,\frac{\mu}{2}}+ig[A_{n,\frac{\mu}{2}},A_{n,\frac{\nu}{2}}])\} \\ &\sim exp\{i\frac{a^2g}{4}F_{n,\mu\nu}\}, \end{aligned} \tag{32}$$

with the definition

$$F_{n,\mu\nu} = \partial_\mu A_{n,\frac{\nu}{2}} - \partial_\nu A_{n,\frac{\mu}{2}} + ig[A_{n,\frac{\mu}{2}}, A_{n,\frac{\nu}{2}}], \tag{33}$$

Then we discuss the continuum form of the kinematic term of the Fermion Lagrangian

$$\bar{\psi}_n\gamma_\mu[U_{n,\frac{\mu}{2}}R_{\frac{\mu}{2}} - U_{n,-\frac{\mu}{2}}R_{-\frac{\mu}{2}}]\psi_n \tag{34}$$

Expanding this equation to the first order of a, we get

$$\begin{aligned} &\bar{\psi}_n\gamma_\mu[(1+i\frac{a}{2}gA_{n,\frac{\mu}{2}})(\psi_n+\frac{a}{2}\partial_\mu\psi_n) \\ &-(1-i\frac{a}{2}g(A_{n,\frac{\mu}{2}}-\frac{a}{2}\partial_\mu A_{n,\frac{\mu}{2}}))(\psi_n-\frac{a}{2}\partial_\mu\psi_n)] \\ \approx\ &a\bar{\psi}_n\gamma_\mu(\partial_\mu+igA_{n,\frac{\mu}{2}})\psi_n \end{aligned} \tag{35}$$

It should be pointed out that, in our scheme, scalar and fermion fields can be dealt with in an unified manner. Instead of taking the lattice gradient operator as in Eq (9), we can also use the new operator in (16) for scalar field. Then the kinematic part of the Boson Lagrangian should be written as

$$[(R_{\frac{\mu}{2}} - R_{-\frac{\mu}{2}})\phi_n]^+[(R_{\frac{\mu}{2}} - R_{-\frac{\mu}{2}})\phi_n]$$

which, when transfering to the momentum space, leads to the exactly same formula as in Eq (4). This means that the dispersion relation (5) does not change. Now if we want to write the gauge invariant version of the scalar action (1), the above kinamatic term has to be changed into

$$[(U_{n,\frac{\mu}{2}}R_{\frac{\mu}{2}} - U_{n,-\frac{\mu}{2}}R_{-\frac{\mu}{2}})\phi_n]^+[(U_{n,\frac{\mu}{2}}R_{\frac{\mu}{2}} - U_{n,-\frac{\mu}{2}}R_{-\frac{\mu}{2}})\phi_n] \tag{36}$$

Using (26) (19) (21) it is easy to prove the gauge invariance of (36). We also write down the form of (36) in the continuum limit as

$$\begin{aligned}
&(U_{n,\frac{\mu}{2}}R_{\frac{\mu}{2}} - U_{n,-\frac{\mu}{2}}R_{-\frac{\mu}{2}})\phi_n \\
&\sim (1 + i\frac{a}{2}gA_{n,\frac{\mu}{2}})(\phi_n + \frac{a}{2}\partial_\mu\phi_n) - (1 - i\frac{a}{2}g(A_{n,\frac{\mu}{2}} - \frac{a}{2}\partial_\mu A_{n,\frac{\mu}{2}}))(\phi_n - \frac{a}{2}\partial_\mu\phi_n) \\
&\sim a(\partial_\mu + igA_{n,\frac{\mu}{2}})\phi_n
\end{aligned} \tag{37}$$

Now we can use above results to construct the Weinberg-Salam model in lattice theory as an illustration. This can be done straightforward by adding (33) (12) (24) (25) and other common terms together. In W-S model, there are two kinds of gauge groups: $SU(2)$ and $U(1)$, which we denotes as :

$$U_{n,\frac{\mu}{2}} = e^{i\frac{a}{2}gA_{n,\frac{\mu}{2}}} = e^{i\frac{a}{2}g\frac{\lambda^i}{2}A^i_{n,\frac{\mu}{2}}}, \qquad U \in SU(2), \tag{38}$$

$$V_{n,\frac{\mu}{2}} = e^{-i\frac{a}{2}g'B_{n,\frac{\mu}{2}}}, \quad V \in U(1). \tag{39}$$

Therefore the total lattice action is written as:

$$S_{W-S} = S(\psi) + S(A) + S(B) + S(H) + S(Yukawa) \tag{40}$$

where

$$S(\psi) = \sum_\alpha \sum_n \{a^3 \sum_\mu \bar{\psi}^\alpha_n \gamma_\mu [U_{n,\frac{\mu}{2}}V_{n,\frac{\mu}{2}}R_{\frac{\mu}{2}} - U_{n,\frac{-\mu}{2}}V_{n,\frac{-\mu}{2}}R_{\frac{-\mu}{2}}]\psi^\alpha_n\} \tag{41}$$

(here α denotes different kinds of Fermions: left-hand-doublets and right-hand-singlete of leptons and quarks in three generations.)

$$S(A)+S(B)=\frac{-8}{g^2}\sum_p trU_p+\frac{-8}{g'^2}\sum_p trV_p, \quad p\in \quad all \quad plaquettes, \tag{42}$$

$$\begin{aligned} S(H)=\sum_n\{a^2\sum_\mu[(U_{n,\frac{\mu}{2}}V_{n,\frac{\mu}{2}}R_{\frac{\mu}{2}}-U_{n,\frac{-\mu}{2}}V_{n,\frac{-\mu}{2}}R_{\frac{-\mu}{2}})\phi_n]^+[(U_{n,\frac{\mu}{2}}V_{n,\frac{\mu}{2}}R_{\frac{\mu}{2}} \\ -U_{n,\frac{-\mu}{2}}V_{n,\frac{-\mu}{2}}R_{\frac{-\mu}{2}})\phi_n]+a^4[\mu^2\phi_n^+\phi_n+\lambda(\phi_n^+\phi_n)^2]\}, \end{aligned} \tag{43}$$

(here ϕ is the Higgs doublets.) and

$$\begin{aligned} S(Yukawa)=\sum_n a^4\{[\bar{l}_n^i\phi_n M_{ij,l}e_n^j+h.c] \\ +[\bar{q}_{n,L}^i\tilde{\phi}_n M_{ij,up}u_n^j+h.c] \\ +[\bar{q}_{n,L}^i\phi_n M_{ij,down}d_n^j+h.c]\} \end{aligned} \tag{44}$$

(here i,j denote the three generations.) It can be shown that, with the helps of Eqs(32) (35) and(37), the action (40) leads to the ordinary standard Weinberg-Salam model action in the continuum limit.

In summary, in this paper we have proposed the replacement of $\partial_\mu \longrightarrow \frac{R_{\frac{\mu}{2}}-R_{\frac{-\mu}{2}}}{a}$ in lattice theory. By doing so, we can eliminate the Fermion-doubling problem, get the same dispersion relation for both Boson and Fermion fields and preserve the chiral symmetry. We can also construct the gauge theory in new frame. Because of these good characters, we can discuss chiral model which is difficult to touch upon in normal lattice theory. Furthermore we can construct supersymmetry lattice model in this approach. There will be many interesting problems need to be considered further and those will be our next works.

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